

JOURNAL OF ALGEBRA **83**, 1–5 (1983)

## Canonical Modules of Partially Ordered Sets

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Received July 29, 1981

In this brief note we show that the canonical module of the Stanley–Reisner ring of a doubly Cohen–Macaulay ordered set is isomorphic to a certain ideal of the same ring. For a general finite partially ordered set, the corresponding ideal is isomorphic to a submodule of the canonical module. For an introduction to Cohen–Macaulay ordered sets from the ring-theoretical point of view see Garsia [4] and Baclawski–Garsia [3]. Doubly Cohen–Macaulay ordered sets were introduced in Baclawski [2].

Let  $\Delta$  be a finite simplicial complex of rank  $r$  (dimension  $r - 1$ ) on vertex set  $V$ . We write  $\Delta_k$  for  $\{\sigma \in \Delta \mid |\sigma| = k\}$ . We assume that  $\Delta$  can be *colored*, i.e., there exists a map  $c: V \rightarrow [r] = \{1, 2, \dots, r\}$  such that for every  $\sigma \in \Delta$ ,  $|c(\sigma)| = |\sigma|$ ; and we fix a choice of coloring henceforth. The most important example of a colored complex is the simplicial complex  $\Delta(P)$  of chains of a partially ordered set (poset)  $P$ . For example, the map  $c: P \rightarrow [r]$  given by

$$c(x) = \max\{k \mid \text{there exists } x_1, \dots, x_{k-1} \in P \\ \text{such that } x_1 < x_2 < \dots < x_{k-1} < x\},$$

is a coloring of  $\Delta(P)$ .

If we regard each element of  $V$  as an indeterminate, then the Stanley–Reisner ring is the quotient ring

$$K[\Delta] = K[v \mid v \in V] / (v_1 \cdots v_k \mid \{v_1, \dots, v_k\} \notin \Delta).$$

Let  $\theta_i \in K[\Delta]$  be defined by

$$\theta_i = \sum_{v \in V} v \chi(c(v) = i),$$

where  $\chi(\mathcal{P})$  is 1 if  $\mathcal{P}$  is true and 0, otherwise. We call  $\theta_i$  the  *$i$ th rank sum* of  $K[\Delta]$ . The sequence  $\theta_1, \dots, \theta_r$  is a homogeneous system of parameters (*frame*)

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of  $K[\Delta]$ . We write  $K[\theta]$  for  $K[\theta_1, \dots, \theta_r] \subseteq K[\Delta]$ . The simplicial complex  $\Delta$  is said to be *Cohen–Macaulay* (CM) if  $K[\Delta]$  is a free  $K[\theta]$  module.

A simplicial complex  $\Delta$  is said to be 2-CM *connected* (or *doubly* CM) if for every vertex  $v \in V$ , we have

$$\Delta \setminus \{v\} = \{\sigma \in \Delta \mid v \notin \sigma\} \quad \text{is CM,} \quad (1)$$

$$r(\Delta \setminus \{v\}) = r(\Delta). \quad (2)$$

An important example of such a complex is  $\Delta(P)$ , where  $P = L \setminus \{\hat{0}, \hat{1}\}$  and  $L$  is a finite geometric lattice. For a proof of this and for other examples see [2].

The *canonical module* of a CM graded  $K$  algebra  $R$  is the graded  $R$  module

$$\Omega(R) = \text{Hom}_{K[\alpha]}(R, K[\alpha]),$$

where  $\alpha_1, \dots, \alpha_r$  is any frame of  $R$ . The canonical module is independent of the choice of a frame, see Herzog–Kunz [5, Remark 5.19]. We will use this as the definition of  $\Omega(R)$  even when  $R$  is not CM.

We will use multiindex notation for denoting nonzero monomials of  $K[\Delta]$ . Thus if  $N = (n_1, \dots, n_k)$  is a sequence of nonnegative integers, then  $v^N$  stands for the monomial  $v^N = \prod_{i=1}^k (v_i)^{n_i}$ . We will always assume that the vertices  $v_1, \dots, v_k$  in such a monomial satisfy  $c(v_1) < c(v_2) < \dots < c(v_k)$ . The *rank set*  $r(v^N)$  is the *multiset* for which each color  $c(v_i)$  occurs exactly as often as  $v_i$  occurs as a factor of  $v^N$ . Equivalently we may think of  $r(v^N)$  as the  $r$ -tuple  $\sum_{i=1}^k n_i e(c(v_i))$ , where  $e(j)$  is the  $r$ -tuple whose  $i$ th component is  $\delta_{ij}$ . The vector  $\rho$  is the  $r$ -tuple  $(1, \dots, 1) = \sum_{i=1}^r e(i)$ . The rank set defines a *multigrading* on  $K[\Delta]$ . The *support*,  $\text{supp}(v^N)$ , of the monomial  $v^N$  is the set  $\{v_i \mid n_i > 0\}$ . Recall that an (*algebraic*) *chain* is a formal linear combination of elements of  $\Delta$ . For each  $\sigma \in \Delta$  there is a corresponding monomial  $\prod_{v \in \sigma} v \in K[\Delta]$ . Thus we may regard the space of chains as a subspace of  $K[\Delta]$ . In particular the  $(r-1)$ th reduced homology  $\tilde{H}_{r-1}(\Delta, K)$ , being a subspace of the space of chains of rank  $r$ , is a subspace of  $K[\Delta]$ . Write  $J(\Delta)$  for the ideal of  $K[\Delta]$  generated by  $\tilde{H}_{r-1}(\Delta, K)$ .

Define a linear map  $\varphi: K[\Delta] \rightarrow K[\theta]$  as

$$\begin{aligned} \varphi(v^N) &= \theta_1^{n_1-2} \dots \theta_r^{n_r-2}, & \text{if } r(v^N) \geq 2\rho, \\ &= 0, & \text{otherwise.} \end{aligned}$$

The inequality  $r(v^N) \geq 2\rho$  is componentwise. Note that  $k=r$  when this inequality holds. Now  $\varphi$  is a multihomogeneous linear map of degree  $-2r$ ,

which is generally not a  $K[\theta]$  homomorphism. We use the map  $\varphi$  to define a homomorphism

$$\psi: J(\Delta) \rightarrow \text{Hom}_K(K[\Delta], K[\theta]),$$

as follows: if  $g \in J(\Delta)$  and  $f \in K[\Delta]$ , then let  $\psi(g)(f)$  be  $\varphi(gf)$ . It is easy to see that  $\psi$  is a  $K[\Delta]$  homomorphism. We first show that  $\psi$  is injective; no special property of  $J(\Delta)$  is used for this. Next we prove that  $\text{Im } \psi$  is contained in  $\Omega(K[\Delta])$ . Finally, we prove that  $\text{Im } \psi = \Omega(K[\Delta])$  when  $\Delta$  is doubly CM.

LEMMA.  $\psi$  is injective.

*Proof.* Suppose that  $\psi(g) = 0$ . We may assume that  $g$  is multihomogeneous. Note that every term of  $g$  is supported on a simplex  $\sigma \in \Delta_r$ . Let  $av^N$  be one term of  $g$ . Then  $0 = \psi(g)(v^\rho) = \varphi(gv^\rho) = \varphi(av^{N+\rho}) = a\theta^{N-\rho}$ . Thus  $a = 0$ . We may do the same for any term of  $g$ . Hence  $g = 0$ .

Q.E.D.

We now come to our first main result.

THEOREM 1. For any colored complex  $\Delta$ , there is a monomorphism

$$\psi: J(\Delta) \rightarrow \Omega(K[\Delta]).$$

*Proof.* We wish to show that  $\text{Im } \psi \subseteq \text{Hom}_{K[\theta]}(K[\Delta], K[\theta])$ . Clearly it suffices to show that  $\varphi(\theta_i g) = \theta_i \varphi(g)$  for any  $i$  and any  $g \in J(\Delta)$ . Moreover we may assume that  $g$  is multihomogeneous.

Consider first the case  $r(g) \geq 2\rho$ . If  $g$  were simply a monomial  $v^N$ , then  $\theta_i g$  would be  $v^{N+e(i)}$  and hence  $\varphi(\theta_i g) = \theta^{N+e(i)-2\rho} = \theta_i \varphi(g)$ . In general  $g$  would be a linear combination of such terms, so the result follows by linearity. Next suppose that  $e(i) + r(g) \not\geq 2\rho$ . Then  $\varphi(\theta_i g) = 0$  and  $\varphi(g) = 0$  so this case also follows.

There remains the case for which  $r(g) \not\geq 2\rho$  but  $e(i) + r(g) \geq 2\rho$ . Each term of  $g$  has the form  $av^N$ , where  $N = (n_1, \dots, n_r)$  satisfies  $n_i = 1$  and  $n_j \geq 2$  for  $j \neq i$ . By definition of  $J(\Delta)$  we may find  $g_j \in K[\Delta]$  and  $f_j \in \tilde{H}_{r-1}(\Delta, K)$  such that  $g = \sum_j g_j f_j$ . Now  $r(f_j) = \rho$  so the support of every term of  $g$  is a simplex in  $\Delta_{r-1}$ . Thus we may write

$$f_j = \sum_{\sigma \in \Delta_r} c_j(\sigma) \prod \sigma, \quad g_j = \sum_{\tau \in \Delta_{r-1}} d_j(\tau) h(\tau),$$

where  $c_j(\sigma)$  and  $d_j(\tau)$  are in  $K$ , and where  $h(\tau)$  is the unique monomial such that  $r(h(\tau)) = r(g_j)$  and  $\text{supp}(h(\tau)) = \tau$ . We can then compute

$$g = \sum_j g_j f_j = \sum_j \sum_{\tau \in \Delta_{r-1}} \sum_{\sigma \in \Delta_r} d_j(\tau) c_j(\sigma) h(\tau) \prod \sigma.$$

Now  $h(\tau) \prod \sigma = 0$  unless  $\tau \subset \sigma$ . Thus

$$g = \sum_j \sum_{\sigma, \tau} c_j(\sigma) d_j(\tau) h(\tau) \prod \sigma \chi(\sigma \supset \tau).$$

Next we use the fact that  $f_j \in \tilde{H}_{r-1}(\Delta, K) = \text{Ker}(\partial)$ . Thus

$$\begin{aligned} 0 &= \partial f_j = \partial \sum_{\sigma} c_j(\sigma) \prod \sigma \\ &= \sum_{\sigma} c_j(\sigma) \sum_{\tau \in \Delta_{r-1}} (-1)^{c(\sigma \setminus \tau)} \prod \tau \chi(\tau \subset \sigma) \\ &= \sum_{\tau \in \Delta_{r-1}} \left( \sum_{\sigma \supset \tau} (-1)^{c(\sigma \setminus \tau)} c_j(\sigma) \right) \prod \tau. \end{aligned}$$

Since  $c(\sigma \setminus \tau) = r - c(\tau)$  depends only on  $\tau$ , it follows that for every  $\tau \in \Delta_{r-1}$ , we have  $\sum_{\sigma \supset \tau} c_j(\sigma) = 0$ . Now  $\varphi(\theta_i h(\tau) \prod \sigma) = \theta^{r(g) + e(i) - 2\rho}$  is the same for every  $\sigma \in \Delta_r$  and  $\tau \in \Delta_{r-1}$ . Thus

$$\varphi(\theta_i g) = \sum_j \sum_{\tau} \left( \sum_{\sigma \supset \tau} c_j(\sigma) \right) d_j(\tau) \theta^{r(g) + e(i) - 2\rho} = 0.$$

Since  $\varphi(g) = 0$ , it follows that  $\varphi(\theta_i g) = \theta_i \varphi(g)$ , so  $\varphi$  is a  $K[\theta]$  homomorphism in all cases. Q.E.D.

We now invoke a characterization of the 2-CM property in terms of resolutions in order to obtain

**THEOREM 2.** *Let  $\Delta$  be a colored, CM complex. Then  $\Delta$  is 2 CM if and only if  $\psi: J(\Delta) \rightarrow \Omega(K[\Delta])$  is an isomorphism.*

*Proof.* Since  $\Delta$  is CM,  $|\mu(\Delta)| = \tilde{h}_{r-1}(\Delta, K) = \dim_K \tilde{H}_{r-1}(\Delta, K)$ . Now if  $\psi$  is an isomorphism, then  $\Omega(K[\Delta])$  is generated by  $|\mu(\Delta)|$  elements. By [2, Corollary 4.7],  $\Delta$  is 2 CM.

Conversely, in the proof of [2, Theorem 4.5] it is shown that if  $\Delta$  is 2 CM, then the minimum number of generators of  $\Omega(K[\Delta])$  is  $|\mu(\Delta)|$ , and moreover, all of the generators in a minimum set have the same degree. See also the discussion in [1, Sect. 7]. Since  $\Omega(K[\Delta]) \neq 0$  this implies that  $\mu(P) \neq 0$  and hence that  $\tilde{H}_{r-1}(\Delta, K) \neq 0$ . By standard results in commutative algebra (see [3, Proposition 2.3(3)]),  $K[\Delta]$  is a free  $K[\theta]$  module with a basis given by any linear basis of  $K[\Delta]/(\theta)$ . By [3, Theorem 5.1], the highest degree of a homogeneous element of  $K[\Delta]/(\theta)$  is *exactly*  $r$ , since  $\tilde{H}_{r-1}(\Delta, K) \neq 0$ . Thus the lowest degree of an element of  $\Omega(K[\Delta])$  is  $-r$ . Now  $\psi$  has degree  $-2r$  so the generators of  $J(\Delta)$  get mapped to elements of  $\Omega(K[\Delta])$  of degree  $-r$ . Since  $\psi$  is injective by Theorem 1, and since  $\Omega(K[\Delta])$  has  $|\mu(\Delta)| = \tilde{h}_{r-1}(\Delta, K)$  generators all of the same degree, it follows that  $\psi$  is also surjective. Q.E.D.

*Remark.* It is clear that  $\varphi$  allows us to define an injective map  $\psi: K[\Delta] \rightarrow \text{Hom}_K(K[\Delta], K[\theta])$  for any pure, colorable, simplicial complex. Thus  $\Omega(K[\Delta])$  is isomorphic to an ideal of  $K[\Delta]$  whenever  $\text{Im } \psi$  contains  $\Omega(K[\Delta])$ . It would be interesting to find conditions under which this holds.

#### ACKNOWLEDGMENT

The author would like to thank the Institut Mittag-Leffler for their hospitality while this manuscript was being prepared.

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